

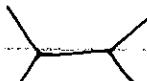
4-point vertex

So far we've studied the 2-point function, & found that we can write the exact propagator as

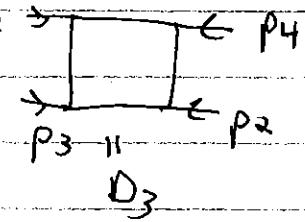
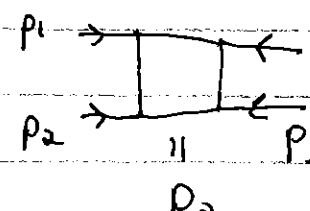
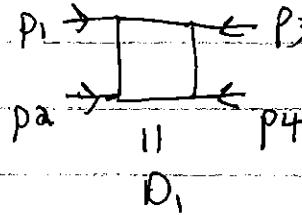
$$\frac{i}{k^2 - m^2 - \Pi(k^2)}, \text{ where } -i\Pi = \sum \text{1PI diagrams}$$

We've also studied $iV_3 = \sum \text{1PI contributions to 3-point function}$

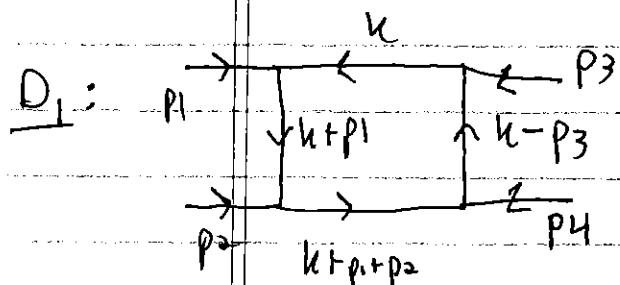
Both were divergent, & required renormalization \Rightarrow this allowed us to Fix Z_ϕ, Z_m, Z_g . Let's continue & study iV_4 , the 1PI contributions to the 4-point Function.

Note that diagrams such as  don't contribute

Label the external momenta as $p_1, p_2, p_3, p_4 \Rightarrow$ all incoming 3 diagrams: $p_1 \rightarrow$



Note that we need only compute one of these, say D_1 ; to get D_2 , take $p_3 \leftrightarrow p_4$ in D_1 ; to get D_3 , take $p_2 \rightarrow p_3$, $p_3 \rightarrow p_4$, $p_4 \rightarrow p_2$



$$iV_4^{(0)} = (ig)^4 i^4 \int \frac{d^d k}{(2\pi)^d} \frac{1}{[(k+p_1)^2 - m^2]} \frac{1}{[(k+p_1+p_2)^2 - m^2]} \frac{1}{[(k-p_3)^2 - m^2]}$$

By now we know how to deal with this. Combine denominators using $\frac{1}{ABCD} = \int_0^1 dx_1 dx_2 dx_3 dx_4 \frac{1}{\{x_1 A + x_2 B + x_3 C + x_4 D\}^4} \delta(1 - x_1 - x_2 - x_3 - x_4)$

$$= 6 \int dF_4 \frac{1}{\{x_1 A + x_2 B + x_3 C + x_4 D\}^4}$$

Upon doing so, the denominator becomes

$$k^2 - m^2 + 2x_1 k \cdot p_1 + x_1 p_1^2 + 2x_2 k \cdot p_1 + 2x_2 k \cdot p_2 + x_2 S - 2x_3 k \cdot p_3 + x_3 p_3^2 \\ \uparrow (p_1 + p_2)^2$$

Shift $k = k' - (x_1 + x_2)p_1 - x_2 p_2 + x_3 p_3$

$$k^2 = k'^2 - 2(x_1 + x_2)k' \cdot p_1 - 2x_2 k' \cdot p_2 + 2x_3 k' \cdot p_3 + (x_1 + x_2)^2 p_1^2 \\ + x_2^2 p_2^2 + x_3^2 p_3^2 + 2x_2(x_1 + x_2)p_1 \cdot p_2 - 2x_3(x_1 + x_2)p_1 \cdot p_3 \\ - 2x_2 x_3 p_2 \cdot p_3$$

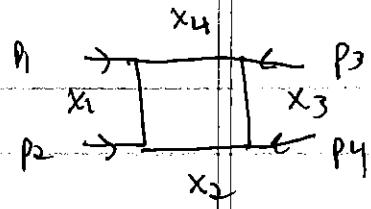
$$k \cdot p_1 \rightarrow k' \cdot p_1 - (x_1 + x_2)p_1^2 - x_2 p_1 \cdot p_2 - x_3 p_1 \cdot p_3$$

$$k \cdot p_2 \rightarrow k' \cdot p_2 - (x_1 + x_2)p_1 \cdot p_2 - x_2 p_2^2 + x_3 p_2 \cdot p_3$$

$$k \cdot p_3 \rightarrow k' \cdot p_3 - (x_1 + x_2)p_1 \cdot p_3 - x_2 p_2 \cdot p_3 + x_3 p_3^2$$

Sum these to find $k^2 - m^2 + x_1 x_4 p_1^2 + x_1 x_2 p_2^2 + x_2 x_3 p_4^2 + x_3 x_4 p_3^2 \\ + x_2 x_4 (p_1 + p_2)^2 + x_1 x_3 (p_1 + p_3)^2 \\ = k^2 - \Delta_{1234} \quad \text{Δ denotes momenta ordering}$

Check the algebra; then note the easy way to immediately write this down



Each pair $x_i x_j$ multiplies the sum of external momenta between the two propagators corresponding to $x_i x_j$

$$\Rightarrow i V_4^{(0)} = 6g^4 \left\{ dF_4 \left(\frac{d^d k}{(2\pi)^d} \frac{1}{\epsilon k^2 - \Delta_{1234}} \right) \right\}^4$$

Now, note the difference between this & previous calculations. Imagine doing the Wick rotation, & then doing the radial integrals in $d=6$

$\int dk \frac{k^5}{k^8} \Rightarrow$ converges! That's good, because we have no more counterterms

Do the k integral using $\int \frac{d^d k}{(2\pi)^d} \frac{1}{\epsilon k^2 - \Delta_{1234}} \right\}^4$

$$= \frac{i}{(4\pi)^3} \frac{\Gamma(1)}{6} \langle \Delta_{1234} \rangle^{-1}$$

$$\Rightarrow i V_4^{(A)} = \frac{i g^4}{64\pi^3} \int dF_4 \frac{1}{\Delta_{1234}}$$

Sum other diagrams by permuting momenta in denom:

$$i V_4 = \frac{i g^4}{64\pi^3} \int dF_4 \left\{ \frac{1}{\Delta_{1234}} + \frac{1}{\Delta_{1243}} + \frac{1}{\Delta_{1342}} \right\}$$

Renormalizability

It turns out that ∇_n for $n \geq 4$ are all finite in ϕ^3 theory in 6-d. The theory is renormalizable: we can absorb the divergences into a Finite number of Lagrangian parameters. If we couldn't do so, the theory would be non-renormalizable.

Let's study a general scalar Field theory & see what ones are renormalizable.

$$\mathcal{L} = \frac{Z_0}{2} \partial_\mu \phi \partial^\mu \phi - \frac{Z_m}{2} m^2 \phi^2 - \sum_{\text{multi}}^{\infty} \frac{1}{n!} Z_n g_n \phi^n$$

Consider an arbitrary loop integral: 

Number of internal propagators: I

Number of loops: L

Count powers of momenta: $D = dL - 2I$

Suppose $D < 0$: then, integral goes like $\frac{d^{dL} k}{k^{dL+2I}}$
 \Rightarrow convergent

If $D = 0$: $\frac{d^{dL} k}{k^{dL}} \Rightarrow \frac{dk}{k}$ \Rightarrow divergent (spherical coords)

$D =$ superficial degree of divergence; if $D \geq 0$, diagram appears to diverge.

Let's see another way to determine D . Let the loop diagram of E external legs \Rightarrow mass dimension of diagram is $[g_E]$

(must have same mass dimension as a tree-level coupling giving rise to a vertex with E legs).

This mass dimension is made up of 3 things: loop integrals, propagators, & the vertices g_n that appear. Suppose the $g_n \phi^n$ vertex appears V_n times. Then, the mass dimension of the diagram is

$$[C_{\text{diagram}}] = \underbrace{dL}_{C_{g_E}} - \underbrace{2I}_{D} + \sum_{n=3}^{\infty} V_n [g_n]$$

C_{g_E} $D \Rightarrow$ solve for it

$$\Rightarrow D = [C_{g_E}] - \sum_{n=3}^{\infty} V_n [g_n]$$

Now suppose we add more & more loops, so that each vertex appears more times. This is no problem if

$[g_n] \geq 0 \Rightarrow D$ doesn't increase. However, if $[g_n] < 0$, diagrams get more divergent. Theories that have a $[g_n] < 0$ are nonrenormalizable. Most fundamental theories of Nature we have (QED, QCD) are renormalizable. The exception is gravity, which is non-renormalizable

$$[G_N] = -2$$

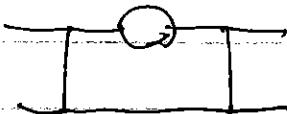
Note that gravity is a fantastically well-tested theory. We can get physical predictions from non-renormalizable theories. The problems associated with having an infinite # of parameters (to remove divergences from more & more loops) only show up typically at high energies. (as the scale associated with the coupling is for gravity;

the Planck mass $\Rightarrow 10^{19}$ GeV). Such non-renormalizable theories are called effective Field theories (effective up to some high energy scale)

There are some subtleties associated with proving renormalizability. Consider the 4-point vertex

$$[g_4] = -2.$$

However,



diverges

The internal loop is called a divergent subdiagram.

In this case, it is cancelled by



The trick is showing that all such divergent subdiagrams have a corresponding counterterm. At the end of the day, the following conclusions are arrived at for theories with particles of the following spins.

Spin 0, $1/2 \Rightarrow$ renormalizable if no couplings $[g_n] < -1$

spin 1 \Rightarrow in addition, a gauge symmetry is required

> spin 1 \Rightarrow never renormalizable in $d \geq 4$

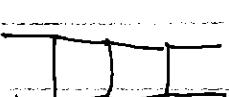
Two-particle scattering at Γ loop

Let's see how to put all of these pieces together to get a full Γ loop scattering amplitude, using $2 \rightarrow 2$ as an example. Define the skeleton expansion: compute the n -point vertex functions $V_n(k_1, \dots, k_n)$, omitting any LPE diagrams containing a subdiagram of 2 or 3 external lines that has a loop.

What does this mean? Look at  \Rightarrow no internal loop

However, consider  \Rightarrow dotted lines indicate a subdiagram of 2 legs that has a loop \Rightarrow omit

Consider  \Rightarrow subdiagram of 3 legs with loop
 \Rightarrow omit

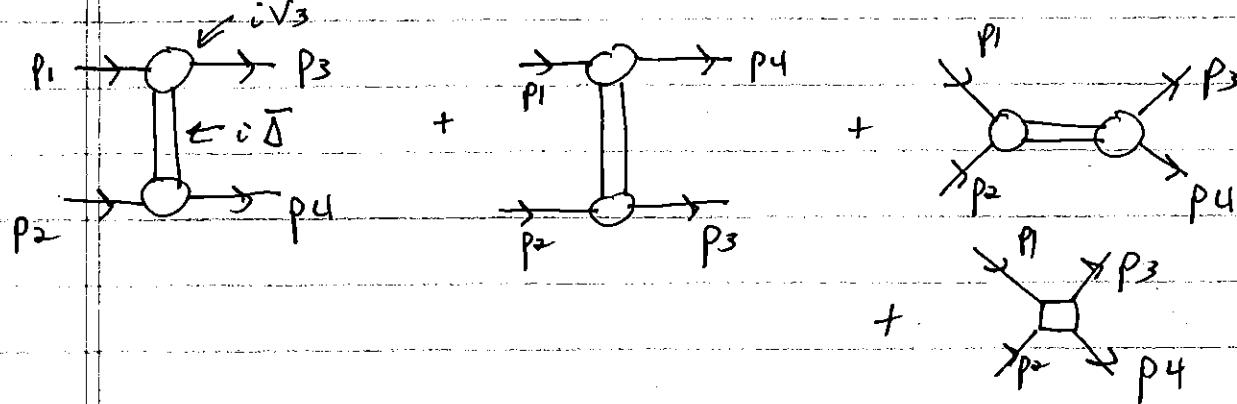
Consider  \Rightarrow no subdiagram of 2, 3 legs with a loop \Rightarrow keep

In each of these skeleton diagrams, use the exact 3-point vertex $\Rightarrow iV_3$ instead of $i\gamma$. For each propagator, use $\frac{i}{k^2 - m^2 - i\Gamma(k^2)} \equiv i\bar{\Delta}(k^2)$

Corrections are organized in this approach such that finite building blocks $V_3, \bar{\Delta}$ are used

Next, draw all tree-level diagrams contributing to the scattering process of interest. In introduce a vertex for iV_3 . For $2 \rightarrow 2$ scattering, for example, would be a 4-particle vertex from iV_4 . For each vertex in the tree-level diagram, use iV_3 . Use $i\bar{J}$ for each propagation.

For $2 \rightarrow 2$ scattering, have



This organizes the calculation into re-usable chunks.

We will study this for $p_i^2 = m^2$, $s, t, u \gg m^2 \Rightarrow$ high-energy scattering limit. To do this, expand iV_3 , $i\bar{J}$, + iV_4 in this limit.

We've already done $V_3: iV_3(p_1^3, p_2^3, s)$

$$\approx ig \left\{ 1 - \frac{g^3}{128\pi^3} \left[\ln\left(\frac{s}{m^2}\right) - i\pi \right] \right\}$$

Similarly, $iV_3(p_1^3, p_3^3, t) \approx ig \left\{ 1 - \frac{g^3}{128\pi^3} \ln\left(\frac{-t}{m^2}\right) \right\}$

positive!

Actually, redo & keep constant term

$$ig \left\{ 1 - \frac{g^2}{64\pi^3} \int_0^1 dx_2 \int_0^{1-x_2} dx_3 \ln \left[-x_2 x_3 \frac{s}{m^2} \right] \right\}$$

$$= -ig \left\{ 1 - \frac{g^2}{64\pi^3} \left[\frac{1}{2} \ln\left(\frac{-s}{m^2}\right) + \underbrace{\int_0^1 dx_2 \int_0^{tx_2} dx_3 \ln[x_2 x_3] \right]}_{-\frac{3}{2}} \right\}$$

$$= ig \left\{ 1 - \frac{g^2}{128\pi^3} \left[\ln\left(\frac{s}{m^2}\right) - i\pi - 3 \right] \right\} \text{ for } V_3(s)$$

while $V_3(t) = ig \left\{ 1 - \frac{g^2}{128\pi^3} \left[\ln\left(\frac{-t}{m^2}\right) - 3 \right] \right\}$

For the propagators, need $\bar{D}(s), \bar{D}(t), \bar{D}(u)$

$$\Rightarrow i\bar{D}(t) = \frac{i}{t - m^2 \Pi(t)} \equiv \frac{i}{t - \Pi(t)}$$

$$\Pi(t) = \frac{-g^2}{128\pi^3} \left\{ \frac{1}{6} (t - m^2) + \int_0^1 dx [m^2 x(t-x) t] \ln \left[\frac{m^2 x(t-x) t}{m^2 (t-x+x^2)} \right] \right\}$$

$$\approx \frac{-g^2}{128\pi^3} \left\{ \frac{1}{6} t - t \int_0^1 dx x(t-x) \ln \left[\frac{-x(t-x) t}{m^2 (t-x+x^2)} \right] \right\}$$

$$= \frac{-g^2 t}{128\pi^3} \left\{ \frac{1}{6} - \frac{1}{6} \ln\left(\frac{-t}{m^2}\right) - \int_0^1 dx x(t-x) \ln \left[\frac{x(t-x)}{t-x+x^2} \right] \right\}$$

$$= \frac{-g^2 t}{128\pi^3} \left\{ -\frac{1}{2} - \frac{1}{6} \ln\left(\frac{-t}{m^2}\right) + \frac{\sqrt{3}\pi}{6} \right\} \quad \frac{2}{3} - \frac{\sqrt{3}\pi}{6}$$

Can write $i\bar{D}(t) \approx \frac{i}{t} \left\{ 1 - \frac{g^2}{128\pi^3} \left[-\frac{1}{2} - \frac{1}{6} \ln\left(\frac{-t}{m^2}\right) + \frac{\sqrt{3}\pi}{6} \right] \right\} + O\left(\frac{m^2}{t}\right)$

Now we need to get

$$i\sqrt{4} = \frac{ig^4}{64\pi^3} \int_0^1 dx_1 \delta(1-x_1) \left\{ \frac{1}{\Delta_{1234}} + \frac{1}{\Delta_{1243}} + \frac{1}{\Delta_{1342}} \right\}$$

$$\text{consider } \Delta_{1234} = m^2 - x_1 x_4 p_1^2 - x_1 x_2 p_2^2 - x_2 x_3 p_4^2 - x_3 x_4 p_3^2 \\ - x_2 x_4 s - x_1 x_3 t - i\delta \\ \cong -x_2 x_4 s - x_1 x_3 t - i\delta$$

$$\Rightarrow \int_0^1 dx_1 \left(\int_0^{1-x_1} dx_2 \left(\int_0^{1-x_1-x_2} dx_3 \frac{1}{-x_2 x_4 s - x_1 x_3 t - i\delta} \right) \right) \rightarrow 1-x_1-x_2-x_3$$

$$\text{set } x_3' = \frac{x_3}{1-x_1-x_2} \Rightarrow dx_3 = \frac{dx_3'}{1-x_1-x_2}$$

$$\Rightarrow \int_0^1 dx_1 \left(\int_0^{1-x_1} dx_2 \left(\int_0^{1-x_1-x_2} dx_3' \frac{1}{-x_2(1-x_3')(1-x_1-x_2)s - x_1 x_3'(1-x_1-x_2)t - i\delta} \right) \right)$$

$$\text{set } x_2' = \frac{x_2}{1-x_1}$$

$$\Rightarrow \int_0^1 dx_1 dx_2 dx_3 \frac{(1-x_1)^2 (1-x_2)}{-x_2(1-x_3)(1-x_2)(1-x_1)s} \left\{ -x_2(1-x_3)(1-x_2)(1-x_1)s \right. \\ \left. - x_1 x_3 (1-x_1)(1-x_2)t - i\delta \right\}$$

$$= \int_0^1 dx_1 dx_2 dx_3 \frac{(1-x_1)}{-x_2(1-x_3)(1-x_1)s - x_1 x_3 t - i\delta}$$

Now this is simple:

$$= -\frac{1}{2} \frac{\pi^2 + \ln^2 \left(\frac{s+i\delta}{t+i\delta} \right)}{s+t} = \frac{1}{2} \frac{\pi^2 + \ln^2 \left[\frac{s+i\delta}{t+i\delta} \right]}{u}$$

Note: $\ln^2 \left[\frac{s+i\delta}{t+i\delta} \right] = \left\{ \ln \left[\frac{s}{t} (1+i\delta) \right] \right\}^2$

$$= \left\{ \ln \left| \frac{s}{t} \right| - i\pi \right\}^2 = \ln^2 \left| \frac{s}{t} \right| - 2i\pi \ln \left| \frac{s}{t} \right| - \pi^2$$

Sum the three contributions to get

$$iV_4 = \frac{i g^4}{128\pi^3} \left\{ \frac{1}{u} \left[\pi^2 + (\ln \left| \frac{s}{t} \right| - i\pi)^2 \right] \right. \\ \left. + \frac{1}{t} \left[\pi^2 + (\ln \left| \frac{s}{u} \right| - i\pi)^2 \right] \right. \\ \left. + \frac{1}{s} \left[\pi^2 + \ln^2 \left(\frac{t}{u} \right) \right] \right\}$$

Okay, put everything together

$$p_1 \rightarrow \text{---} \circ \rightarrow p_3 = iV_3(t) i\bar{\Delta}(t) iV_3(t)$$

$$p_2 \rightarrow \text{---} \circ \rightarrow p_4$$

$$p_1 \rightarrow \text{---} \circ \rightarrow p_4 = i\bar{\Delta}(u) [iV_3(u)]^2$$

$$p_2 \rightarrow \text{---} \circ \rightarrow p_3$$

$$p_1 \rightarrow \text{---} \circ \rightarrow p_3 = i\bar{\Delta}(s) [iV_3(s)]^2$$

$$p_2 \rightarrow \text{---} \circ \rightarrow p_4 = iV_4 \Rightarrow iT = i\bar{\Delta}(t) [iV_3(t)]^2 + i\bar{\Delta}(u) [iV_3(u)]^2 + i\bar{\Delta}(s) [iV_3(s)]^2 + iV_4$$

(a)

Expand to keep through $\mathcal{O}(g^4)$:

$$i^3 \bar{\Delta}(+) [V_3(t)]^2 = \frac{-ig^2}{t} \left\{ 1 + \frac{g^3}{128\pi^3} \left[\frac{1}{2} + \frac{1}{6} \ln\left(-\frac{t}{m^2}\right) + \frac{\sqrt{3}\pi}{6} \right] \right\}$$

$$\left\{ 1 - \frac{g^2}{64\pi^3} \left[\ln\left(-\frac{t}{m^2}\right) - 3 \right] \right\}$$

$$\equiv \frac{-ig^3}{t} \left\{ 1 + \frac{g^2}{128\pi^3} \left[-\frac{11}{6} \ln\left(-\frac{t}{m^2}\right) + \frac{13}{2} + \frac{\sqrt{3}\pi}{6} \right] \right\}$$

$$i^3 \bar{\Delta}(u) [V_3(u)]^2 \equiv \frac{-ig^3}{u} \left\{ 1 + \frac{g^3}{128\pi^3} \left[-\frac{11}{6} \ln\left[-\frac{u}{m^2}\right] + \frac{13}{2} + \frac{\sqrt{3}\pi}{6} \right] \right\}$$

$$i^3 \bar{\Delta}(s) [V_3(s)]^2 \equiv \frac{-ig^2}{s} \left\{ 1 + \frac{g^2}{128\pi^3} \left[\frac{1}{2} + \frac{1}{6} \ln\left(\frac{s}{m^2}\right) - \frac{i\pi}{6} + \frac{\sqrt{3}\pi}{6} \right] \right\}$$

$$\left\{ 1 - \frac{g^2}{64\pi^3} \left[\ln\left(\frac{s}{m^2}\right) - i\pi - 3 \right] \right\}$$

$$\equiv \frac{-ig^2}{s} \left\{ 1 + \frac{g^2}{128\pi^3} \left[-\frac{11}{6} \ln\left(\frac{s}{m^2}\right) + \frac{i11}{6}\pi + \frac{13}{2} + \frac{\sqrt{3}\pi}{6} \right] \right\}$$

$$iV_4 = \frac{-ig^4}{128\pi^3} \left\{ \frac{1}{u} \left[\ln^2\left(\frac{s}{t}\right) - 2i\pi \right] + \frac{1}{t} \left[\ln^2\left(\frac{s}{u}\right) - 2i\pi \ln\left(\frac{s}{u}\right) \right] \right.$$

$$\left. + \frac{1}{s} \left[\pi^2 + \ln^2\left(\frac{t}{u}\right) \right] \right\}$$

$$T = -\frac{g^2}{t} \left[1 + \frac{g^3}{128\pi^3} \left(-\frac{11}{6} \ln\left(-\frac{t}{m^2}\right) + \frac{13}{2} + \frac{\sqrt{3}\pi}{6} - \ln^2\left(\frac{s}{t}\right) + 2i\pi \ln\left(\frac{s}{t}\right) \right. \right.$$

$$\left. \left. - \frac{g^2}{u} \left[1 + \frac{g^2}{128\pi^3} \left\{ -\frac{11}{6} \ln\left(-\frac{u}{m^2}\right) + \frac{13}{2} + \frac{\sqrt{3}\pi}{6} - \ln^2\left(\frac{s}{u}\right) + 2i\pi \ln\left(\frac{s}{u}\right) \right\} \right] \right]$$

$$-\frac{g^2}{s} \left[1 + \frac{g^2}{128\pi^3} \left\{ -\frac{11}{6} \ln\left(\frac{s}{m^2}\right) + \frac{13}{2} + \frac{\sqrt{3}\pi}{6} - \ln^2\left(\frac{t}{u}\right) - \pi^2 + i\frac{11}{6}\pi \right\} \right]$$

as compared to the tree-level value

$$T^{(0)} = -g^2 \left\{ \frac{1}{t} + \frac{1}{u} + \frac{1}{s} \right\}$$

One thing to note: why can't we take $m \rightarrow 0$?